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# Steady-state heat transfer from horizontally insulated slabs

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**Abstract**—A general solution for the steady-state heat conduction problem under a slab-on-grade floor with horizontal insulation is presented. The soil temperature field, the heat flux along the slab, and the total slab heat loss are obtained and analyzed using the Interzone Temperature Profile Estimation (ITPE) technique. The derived solution addresses all the common configurations for horizontal insulation of slab-on-grade floors. The effect of the outer inner edge insulation on heat flux variation along the slab floor surface and on total slab heat loss is discussed and analyzed. Finally, the influence of water table level on total slab heat loss is illustrated for various inner edge insulation configurations.

## 1. INTRODUCTION

THE THERMAL performance of the above-grade portion of buildings has been significantly improved after the energy crisis of the 1970s. As a consequence, the proportional foundation contribution to a building's total heating load has increased. To improve the energy efficiency of slab-on-ground foundations, two insulation configurations are primarily used: (1) vertical insulation placed on the interior or on the exterior of the foundation walls, and (2) horizontal insulation placed under either the slab perimeter or the soil surface outside the slab.

Several models exist for calculating heat losses from uninsulated or uniformly insulated slabs [1–3]. However, very few models have addressed the heat transfer from partially insulated slabs, especially when the insulation extends outside the building. Mitalas [4] provided correlations based on a finite element model for a comprehensive set of slab-on-grade insulation configurations. Unfortunately, the Mitalas correlations are restricted to certain insulation values and limited to particular geometric dimensions. Hagentoft [5] developed a semi-analytical model based on conformal mapping and Fourier series to calculate heat losses from a house with variable thermal insulation thickness along the ground surface. The model did not allow for the existence of a water table underneath the slab.

This paper presents a steady-state solution to the heat conduction problem under slab-on-grade floor with horizontal insulation. The insulation can be placed (i) uniformly under the slab as shown in Fig. 1(a), (ii) a short distance inward from the perimeter of the slab (Fig. 1(b)), or (iii) extending outward from the edge of the slab is indicated in Fig. 1(c). The proposed model can handle any combination of the above mentioned horizontal insulation configurations. A water table effect is considered in this model. The soil temperature field and the heat flux along the slab are obtained and analyzed using the Interzone Temperature Profile Estimation (ITPE) technique [6–10]. The model developed in this paper extends the method for treating slabs developed by Krarti [8]. A parametric discussion is presented on the effect of the outer/inner edge insulation and of the water table level on total slab heat losses. A companion paper will deal with heat loss calculation from a slab-on-grade floor with vertical insulation [11].

## 2. FORMULATION OF THE PROBLEM

Figure 2 shows a model of a slab-on-grade floor with horizontal insulation. The insulation is placed along the perimeter of the slab and can extend along the soil surface. To account for thermal resistance between soil and room air or ambient air, an equivalent air–insulation–slab (if any)–soil conductance, $h$, is introduced:

$$h = (h_o^{-1} + U_i^{-1} + U_s^{-1} + h_i^{-1})^{-1}$$

where

- $h_o$ is a convective heat transfer coefficient above the slab or the soil surface.
- $U_i$ is the insulation conductance.
- $U_s$ is the slab material (or soil layer above insulation extending outward) conductance.
- $h_i$ is the interface contact conductance (slab-to-earth or insulation-to-earth).

The steady-state temperature distribution $T(x, y)$ in the soil beneath the horizontally insulated slab-on-grade floor model of Fig. 2 is subject to the Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (1)$$

with the boundary conditions:

## NOMENCLATURE

| | |
|---|---|
| $a$ | building half width [m] |
| $b$ | water table depth [m] |
| $C_n$, $f_n$ | Fourier coefficients |
| $c$ | interior perimeter insulation location [m] |
| $d$ | exterior perimeter insulation length [m] |
| $H$ | ratio, $h/k_s$ [$m^{-1}$] |
| $H_a$ | ratio, $h_a/k_s$ [$m^{-1}$] |
| $h$ | slab overall heat transfer conductance [$W\ m^{-2}\ K^{-1}$] |
| $h_a$ | exterior perimeter insulation/soil conductance [$W\ m^{-2}\ K^{-1}$] |
| $h_i$ | interface contact (slab–earth or insulation–earth) conductance [$W\ m^{-2}\ K^{-1}$] |
| $h_o$ | convective heat transfer coefficient above the slab or soil surface conductance [$W\ m^{-2}\ K^{-1}$] |
| $k_s$ | soil thermal conductivity [$W\ m^{-1}\ K^{-1}$] |
| $T$ | soil temperature [K] |
| $T_a$ | ambient air temperature [m] |
| $T_i$ | building air temperature [K] |
| $T_s$ | soil surface temperature [K] |
| $T_w$ | water table temperature, assumed to be the reference temperature [K] |
| $U_i$ | insulation conductance [$W\ m^{-2}\ K^{-1}$] |
| $U_s$ | slab or soil layer conductance [$W\ m^{-2}\ K^{-1}$] |
| $x$, $y$ | space coordinates [m]. |

Greek symbols

| | |
|---|---|
| $\alpha_p^f$, $\beta_{n,p}^f$ | coefficients defined in equation (5) |
| $\alpha_p^c$, $\beta_{n,p}^c$, $\gamma_{n,p}^c$ | coefficients defined in equation (7) |
| $\nu_n$, $\mu_n$ | eigenvalues. |

Subscripts

| | |
|---|---|
| a | outer edge of the slab |
| e | inner edge of the slab |
| m | middle of the slab |
| I, II | zone (I), zone (II). |

$$T = 0 \quad \text{for } y = b$$

$$T = T_s \quad \text{for } y = T_w \text{ and } |x| > a+d$$

$$\frac{\partial T}{\partial y} = H(x)(T - T_{ia}(x)) \quad \text{for } y = 0 \text{ and } |x| < a+d$$

where the temperature field has been normalized to $T = 0$ at the water table depth $b$ (i.e. $T_w$ is assumed to be the reference temperature), and where $H(x)$ is the ratio of the equivalent air–insulation–slab–soil conductance to soil thermal conductivity (i.e. $H(x) = h(x)/k_s$). In this paper, $H(x)$ is defined by

$$H(x) = \begin{cases} H_m & \text{if } |x| < c \\ H_e & \text{if } c < |x| < a \\ H_a & \text{if } a < |x| < a+d. \end{cases}$$

Figure 2(b) shows the variation of the function $H(x)$. In Fig. 2(c), the temperature variation $T_{ia}(x)$ is illustrated. $T_i$ is the room air temperature and $T_a$ is the ambient air temperature.

Figure 2(a) shows that the surfaces $|x| = a+d$ divide the ground medium into three zones. Because of the symmetry around the axis $x = 0$, the temperature $T(x, y)$ needs to be determined only in zones (I) and (II). Let $f(y)$ be the temperature profile along the surfaces $|x| = a+d$. Fourier series solution of the Laplace equation (1) can be found using the separation of variables technique. In zones (I), the solution $T_I(x, y)$ is expressed as:

$$T_I(x,y) = \frac{2}{b}\sum_{n=1}^{+\infty} f_n \sin \nu_n y \frac{\cosh \nu_n x}{\cosh \nu_n(a+d)}$$

$$+ \frac{2}{(a+d)}\sum_{n=1}^{+\infty} C_n \cos \mu_n x \frac{\sinh \mu_m(b-y)}{\sinh \mu_n b}. \quad (2)$$

In zone (II), the temperature $T_{II}(x, y)$ is given by

$$T_{II}(x,y) = \frac{2}{b}\sum_{n=1}^{+\infty} \sin \nu_n y \left\{\frac{T_s}{\nu_n}(1 - e^{-\nu_n(|x|-(a+d))}) + f_n\, e^{-\nu_n(|x|-(a+d))}\right\} \quad (3)$$

where,

$$\nu_n = \frac{n\pi}{b}; \quad \mu_n = \frac{(2n-1)}{2(a+d)}\pi.$$

$C_n$ and $f_n$ are Fourier coefficients to be determined.

The continuity of the heat flux at the surface $|x| = a+d$, gives the condition

$$\left.\frac{\partial T_I}{\partial x}\right|_{|x|=a+d} = \left.\frac{\partial T_{II}}{\partial x}\right|_{|x|=a+d} \quad (4)$$

or

$$\frac{2}{b}\sum_{n=1}^{+\infty} \nu_n f_n \tanh(\nu_n(a+d)) \sin \nu_n y - \frac{2}{(a+d)}\sum_{n=1}^{+\infty}(-1)^n \mu_m C_m \frac{\sinh \mu_n(b-y)}{\sinh \mu_n b} = \frac{2}{b}\sum_{n=1}^{+\infty} \sin \nu_n y\{T_s - \nu_n f_n\}.$$

Multiplying the above equation by $\sin \nu_p y$ and integrating over $[0, b]$ yields an expression of the form:

$$f_p = \alpha_p^f + \sum_{n=1}^{+\infty} \beta_{n,p}^f f_n \tag{5}$$

with

$$\alpha_p^f = \frac{T_s}{\nu_p(1+\tanh(\nu_p(a+d)))}$$

$$\beta_{n,p}^f = -\frac{2(-1)^n \mu_n}{(a+d)(1+\tanh(\nu_p(a+d)))(\mu_n^2+\nu_p^2)}.$$

The third-kind boundary condition of equation (1):

$$\left.\frac{\partial T_1}{\partial y}\right|_{y=0} = H(x)(T_1 - T_{ia}(x)) \tag{6}$$

gives the following expression:

$$\frac{2}{b}\sum_{n=1}^{+\infty} \nu_n f_n \frac{\cosh \nu_n x}{\cosh \nu_n(a+d)} - \frac{2}{(a+d)}\sum_{n=1}^{+\infty} \mu_n C_n \coth \mu_n b \cos \mu_n x = H(x)\left\{\frac{2}{(a+d)}\sum_{n=1}^{+\infty} C_n \cos \mu_n x - T_i(x)\right\}.$$

Multiplying this equation by $\cos \mu_p x$ and integrating over the interval $[-a-d, a+d]$ yields:

$$C_p = \alpha_p^c + \sum_{n=1}^{+\infty} \beta_{n,p}^c f_p + \sum \gamma_{n,p}^c C_n \tag{7}$$

with

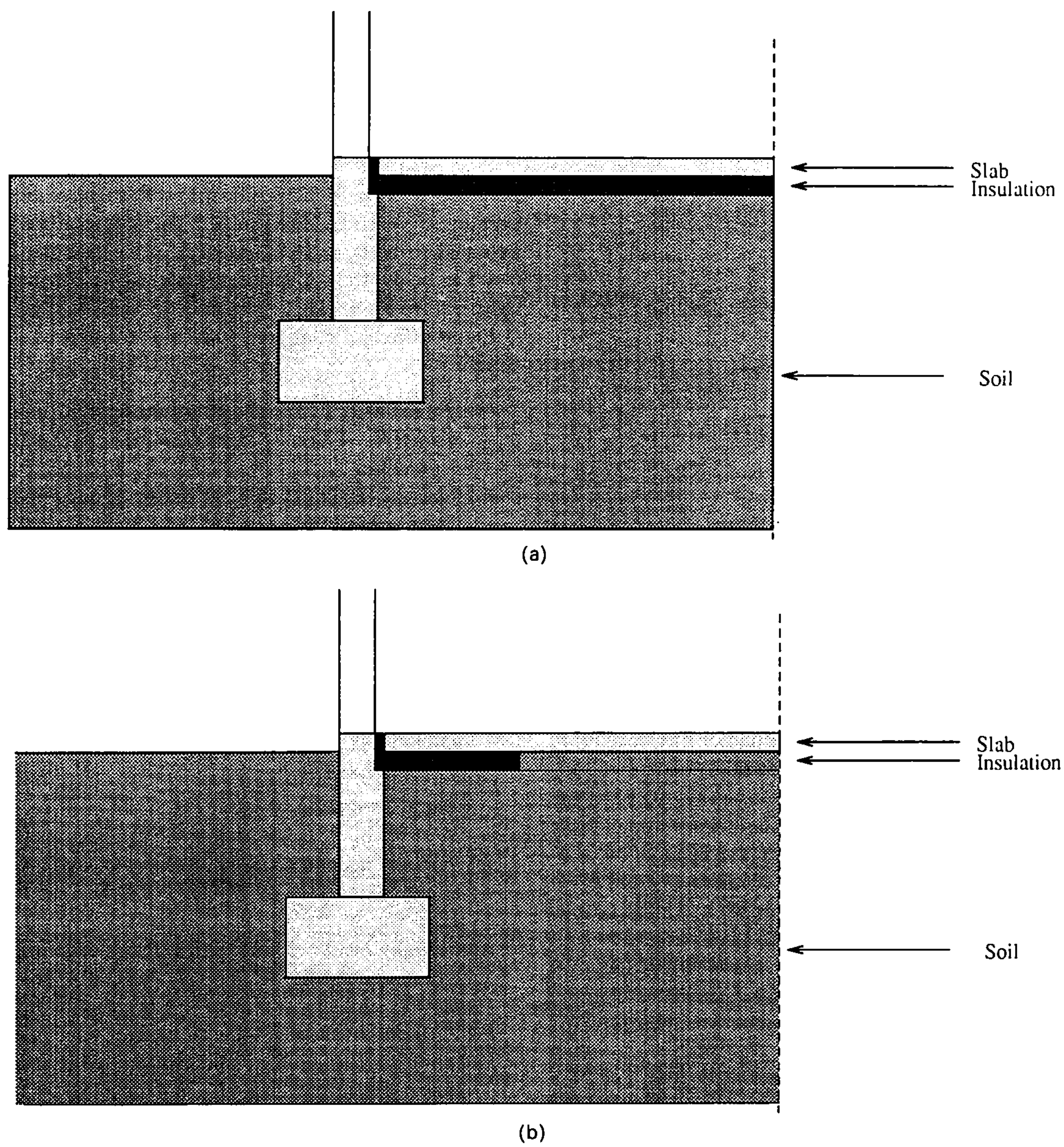
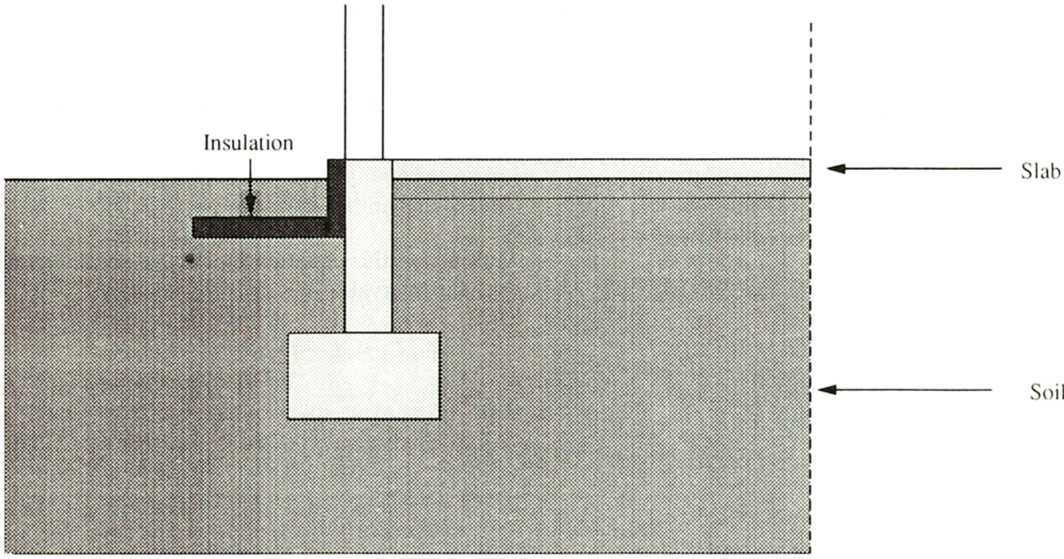



FIG. 1. Horizontal insulation configurations for slab-on-grade floors. (a) Uniform insulation, (b) perimeter insulation under the slab, (c) insulation placed on the building exterior.




(c)

FIG. 1.—*Continued.*

$$\alpha_p^c = \left\{ \frac{2}{\mu_p} T_i [H_e \sin \mu_p a + (H_m - H_e) \sin \mu_p c] - H_a T_a \sin \mu_p a \right\} \Big/ (H_a + \mu_p \coth \mu_p b)$$

$$\beta_{n,p}^c = - \frac{2(-1)^p \mu_p \nu_n}{b(\nu_n^2 + \mu_p)(H_a + \nu_p \coth \mu_p b)}$$

$$\gamma_{n,p}^c = - \frac{2[(H_e - H_a)F_{n,p}^a + (H_m - H_a)F_{n,p}^c]}{(a+d)(H_a + \mu_p \coth \mu_p b)}$$

where

$$F_{n,p}^x = \frac{\sin(\mu_n - \mu_p)x}{(\mu_n - \mu_p)} + \frac{\sin(\mu_n + \mu_p)x}{(\mu_n + \mu_p)}.$$

Note that the case of a partially insulated slab is obtained when $d = 0$ and $H_a = H_e$. Reference [7] treats this particular case in detail.

The system of equations (5) and (7) is solved by truncating the sum to a finite number of terms, $N$. A linear system of $2N$ equations with $2N$ unknowns ($C_p$ and $f_p$, $p = 1, 2, \ldots, N$) is obtained and is solved by the Gauss–Jordan elimination method. The temperatures $T_I(x, y)$ and $T_{II}(x, y)$ are then determined by substituting the values of the coefficients $C_n$ and $f_n$ in equations (2) and (3). For the slab configurations treated in this paper, $N = 25$ gives accurate estimations. Indeed, addition of more terms does not significantly affect the results for $T_I(x, y)$ and $T_{II}(x, y)$ by more than 0.05 K variation in soil temperature, for the cases treated in this paper.

### 3. SOIL TEMPERATURE DISTRIBUTION

Figure 3 shows temperature profiles within soil beneath a slab of width $2a = 10$ m. The air temperature is $T_i = 21°C$, while the soil surface temperature $T_s = 16°C$. The ambient air temperature is assumed to be $T_a = 16°C$. A water table at a depth $b = 5$ m below the soil surface exists with a constant temperature $T_w = 11°C$.

Eight different configurations are considered in Figure 3.

(a) Uninsulated slab without any horizontal insulation ($H_m = 4\ \text{m}^{-1}$; $H_e = H_a = \infty$; $c = d = 0$). Figure 3(a).
(b) Partially insulated slab with interior edge insulation extending 0.5 m ($H_m = 4\ \text{m}^{-1}$, $H_e = 0.1\ \text{m}^{-1}$; $H_a = \infty$; $c = 0.5$ m; $d = 0$ m). Figure 3(b).
(c) Partially insulated slab with interior edge insulation extending 2 m ($H_m = 4\ \text{m}^{-1}$, $H_e = 0.1\ \text{m}^{-1}$; $H_a = \infty$; $c = 2$ m; $d = 0$ m). Figure 3(c).
(d) Uniformly insulated slab ($H_m = H_e = 0.1\ \text{m}^{-1}$; $H_a = \infty$; $c = 5$ m; $d = 0$ m). Figure 3(d).
(e) Partially insulated slab with exterior edge insulation extending 0.5 m ($H_m = H_e = 4\ \text{m}^{-1}$, $H_a = 0.1\ \text{m}^{-1}$; $c = 0$ m; $d = 0.5$ m). Figure 3(e).
(f) Partially insulated slab with exterior edge insulation extending 2 m ($H_m = H_e = 4\ \text{m}^{-1}$; $H_a = 0.1\ \text{m}^{-1}$; $c = 0$ m; $d = 2$ m). Figure 3(f).
(g) Partially insulated slab with exterior edge insulation extending 4 m ($H_m = H_e = 4\ \text{m}^{-1}$, $H_a = 0.1\ \text{m}^{-1}$; $c = 0$ m; $d = 4$ m). Figure 3(g).
(h) Partially insulated slab with interior edge insulation extending 2 m and exterior edge insulation extending 2 m ($H_m = 4\ \text{m}^{-1}$; $H_e = 0.1\ \text{m}^{-1}$; $H_a = 0.1\ \text{m}^{-1}$; $c = 2$ m; $d = 2$ m). Figure 3(h).

Figures 3(a)–(d) show the effect of adding insu-

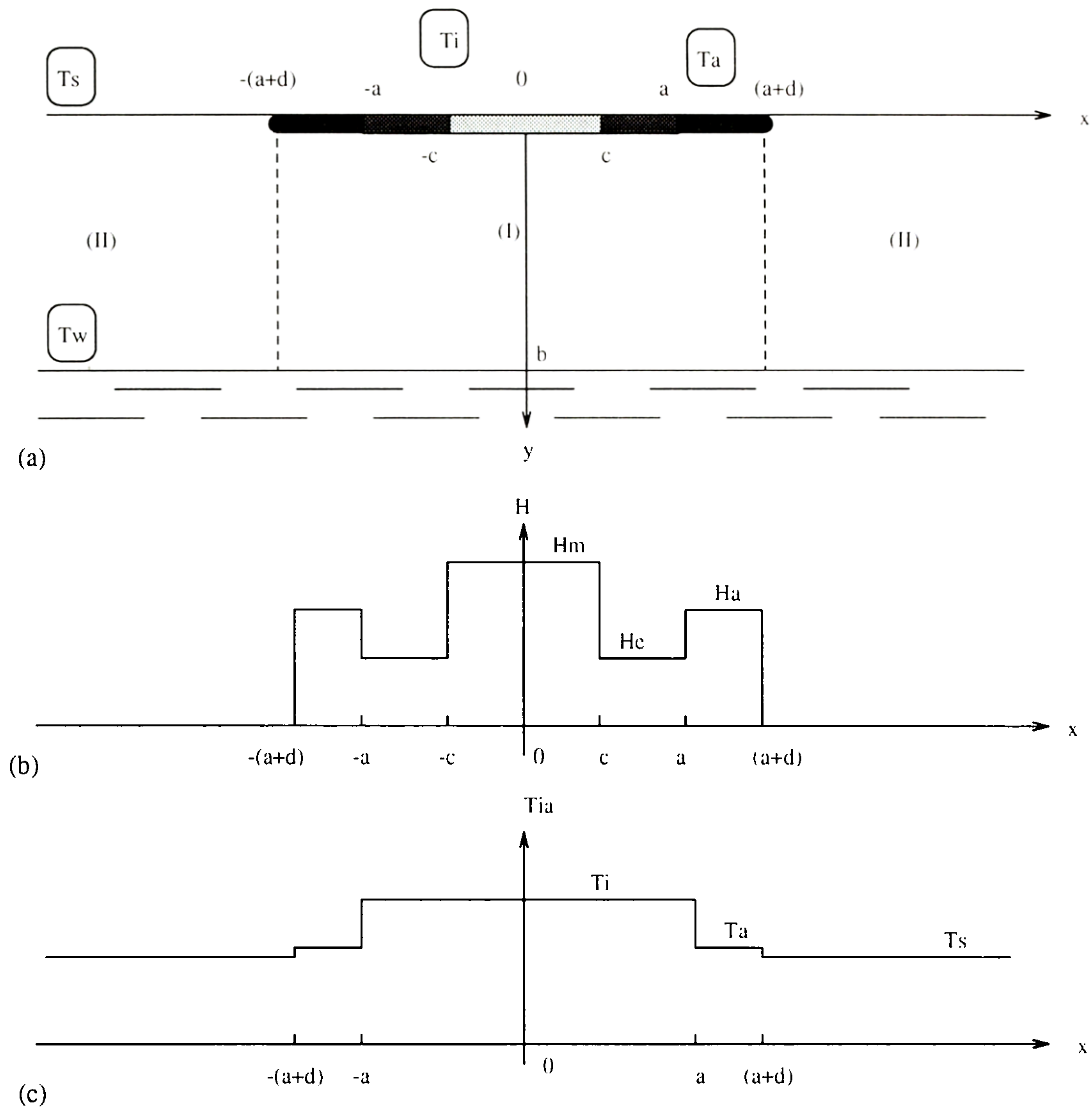



FIG. 2. (a) Slab-on-grade floor with horizontal insulation; (b) distribution of the ratio $H = h/k_s$; (c) distribution of the temperature at the soil surface.

lation on the 'inner' perimeter of an uninsulated slab (i.e. under the slab). The temperature change at the edge of the slab becomes less abrupt as the length of the insulation increases. Indeed, Fig. 3(a) indicates that the temperature varies from 20 to 17°C within only 0.3 m of the perimeter. The same temperature variation occurs over a greater perimeter width when inner insulation is added. This width extends from the slab perimeter to a distance of 0.6 m for 0.5 m insulation, Fig. 3(b), 1.0 m for 2.0 m insulation, Fig. 3(c), and 2 m for a uniform insulation, Fig. 3(d). The decrease in the perimeter slab temperature gradient indicates a decrease in the heat losses from the slab edges. Additional discussion of heat flux from slabs with partial inner perimeter insulation can be found in ref. [8].

Figures 3(e)–(g) show soil temperature distributions when insulation is added horizontally outside the building. The soil surface temperature in the outer vicinity of the slab (but under the insulation) increases with the length of the exterior insulation. In particular the double point (i.e. the point on the soil surface, defined by $-y = 0$ and $|x| > a+d$, and kept at a temperature of 16°C, that meets the 16°C isotherm) moves away from the slab as the exterior insulation length increases. One effect of warming the soil surface around the slab edge is a decrease in heat loss from the slab perimeter as will be discussed later. Note that as the insulation extends still farther from the slab (beyond 2 m), the soil surface temperature reaches a minimum at a distance of 2 m from the slab.

Figure 3(h) illustrates the temperature isotherms beneath a slab insulated with an insulation of approximately $RSI = 10$ (i.e. $H_c = H_a = 0.1\ \mathrm{m}^{-1}$ for a soil with $k_s = 1.0\ \mathrm{W\ m^{-1}\ K^{-1}}$) extending on both exterior and interior edges.

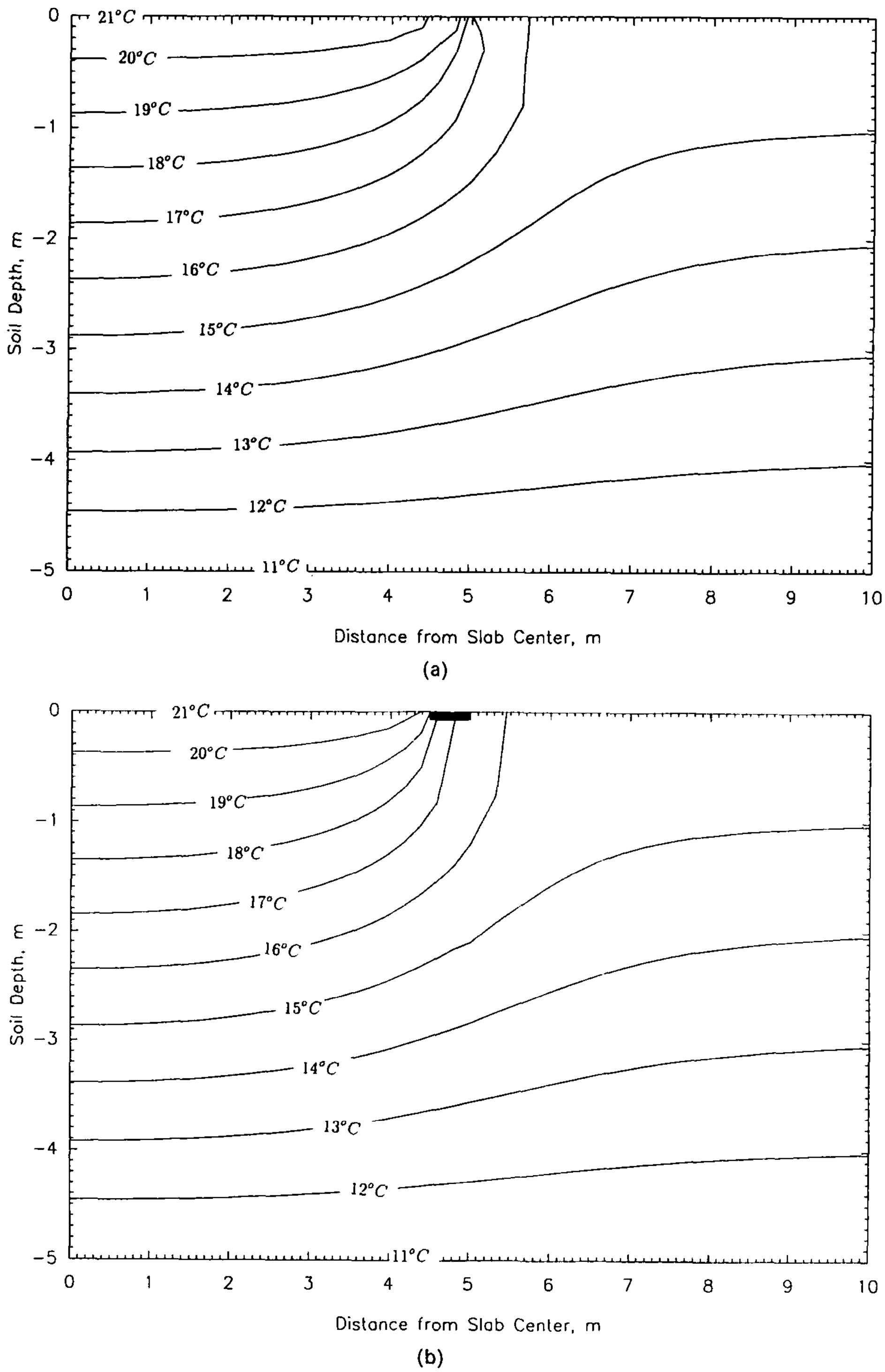



FIG. 3. Earth temperature isotherms beneath a slab-on-grade floor with water table depth $b = 5$ m and (a) $H_m = H_e = 4\ \mathrm{m}^{-1}$, $H_a = \infty$; (b) $H_m = 4\ \mathrm{m}^{-1}$, $H_e = 0.1\ \mathrm{m}^{-1}$, $H_a = \infty$, $c = 0.5$ m.

## 4. HEAT FLUX DISTRIBUTION

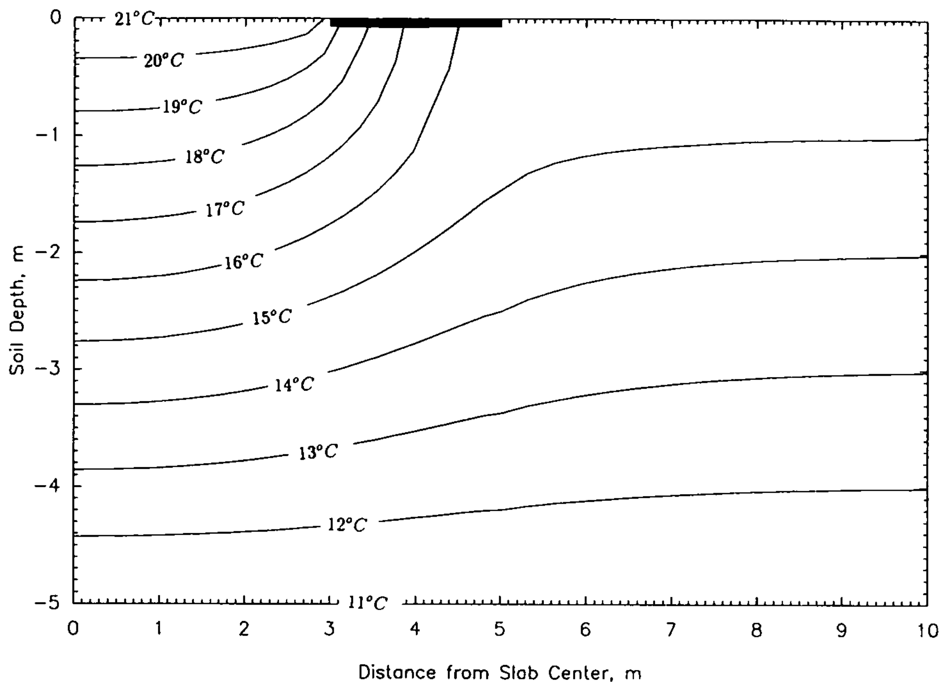
The slab heat flux distribution $q(x)$ is proportional to the temperature gradient along the slab surface

$$q(x) = -k_s \left.\frac{\partial T}{\partial y}\right|_{y=0}. \qquad (8)$$

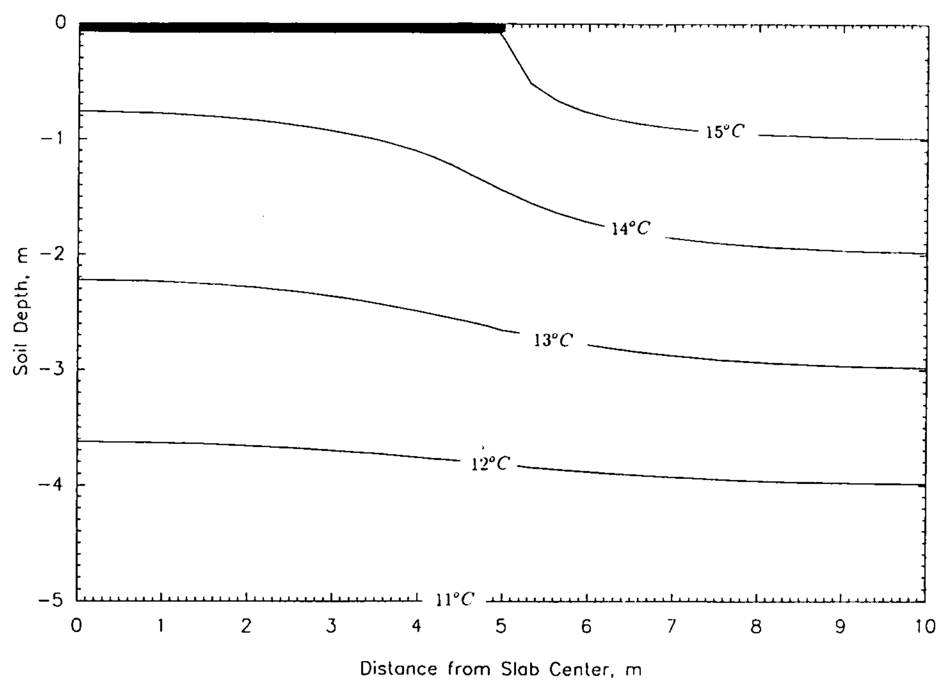
Using the third boundary condition of equation (1), it can be shown that

$$q(x) = -\frac{2}{a} k_s H(x) \sum_{n=1}^{+\infty} \left( C_n + \frac{(-1)^n T_i}{\mu_n} \right) \cos \mu_n x. \qquad (9)$$

Figure 4 shows the heat flux distribution along a




(c)



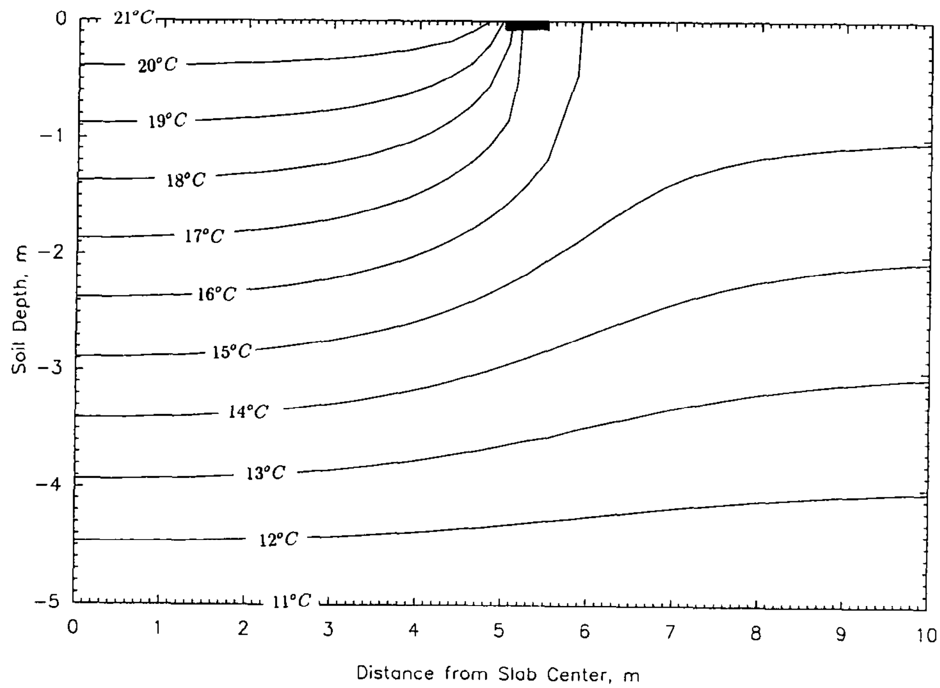

(d)

FIG. 3.—*Continued.* (e) $H_m = H_c = 4\ m^{-1}$, $H_a = 0.1\ m^{-1}$, $d = 0.5$ m; (f) $H_m = H_c = 4\ m^{-1}$; $d = 2$ m.

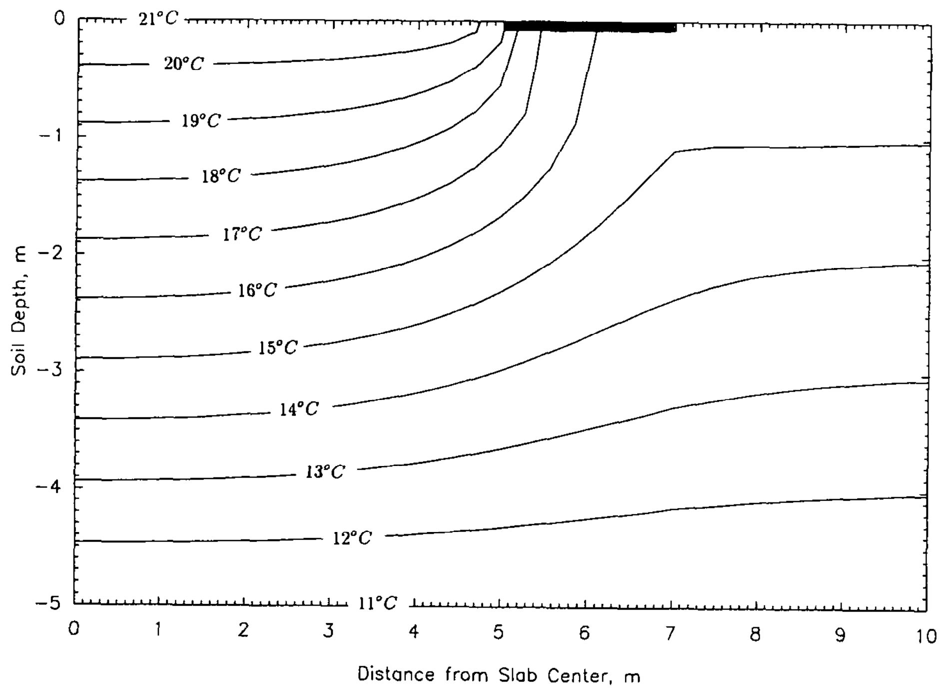
slab of width $2a = 10$ m. The temperature above the slab is $T_i = 21°C$. The soil surface temperature is $T_s = 16°C$, while a water table exists at a depth of $b = 5$ m and is at the temperature $T_w = 11°C$. The case of $H_m = 1\ m^{-1}$ is representative of a moderately insulated slab. Figure 4 indicates clearly that by adding insulation in the outer perimeter of the building, slab heat losses are reduced from the building edges as expected. The heat losses from the central area of the slab are not affected by the presence of perimeter insulation.

## 5. TOTAL SLAB HEAT LOSSES

By integrating the heat flux function $q(x)$ given by equation (9) over the interval $[-a, a]$, the total heat loss from the slab is obtained:




(e)

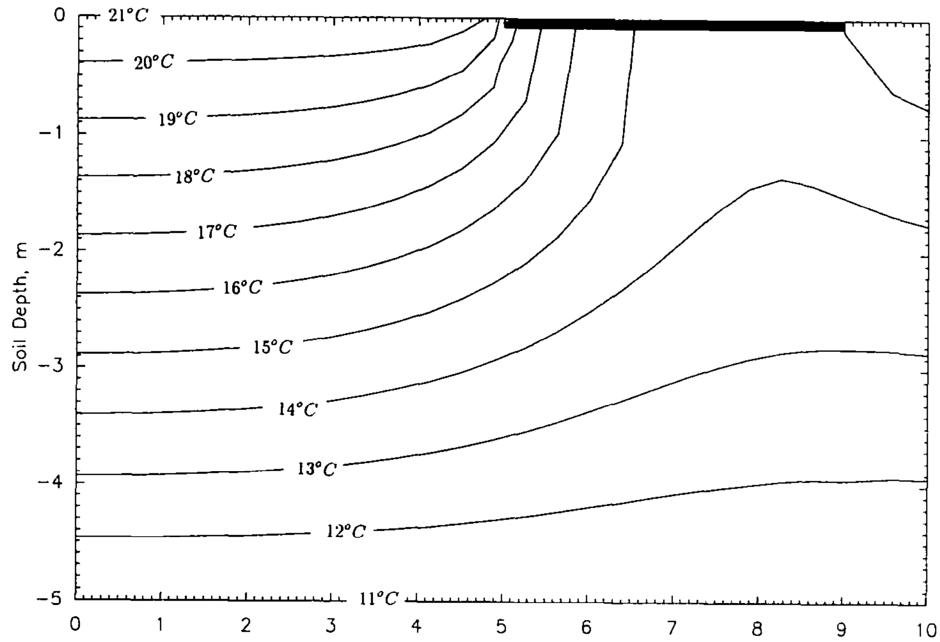



(f)

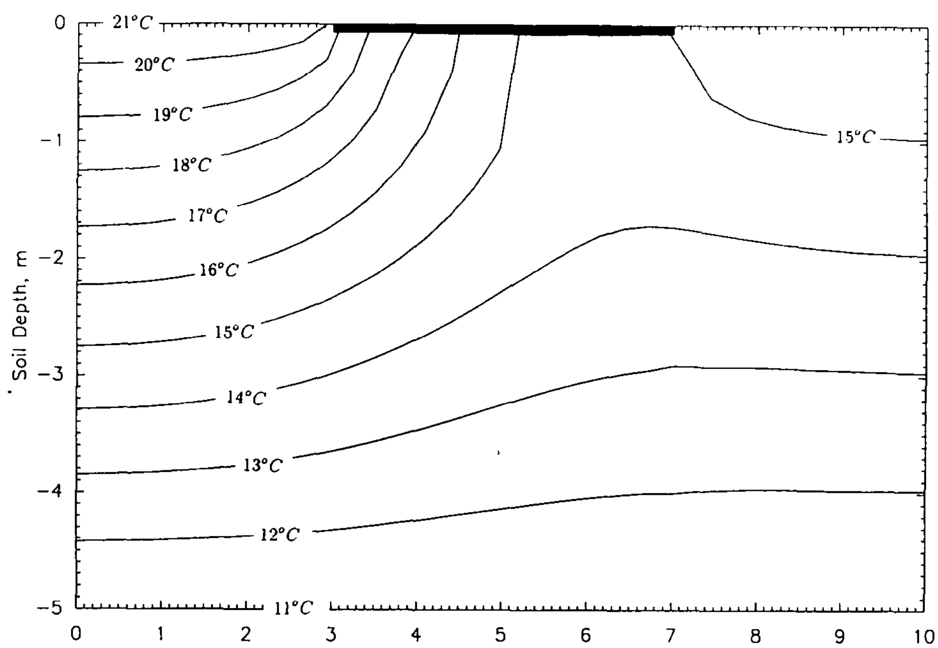
FIG. 3.—*Continued.* (e) $H_m = H_e = 4\ \mathrm{m}^{-1}$, $H_a = 0.1\ \mathrm{m}^{-1}$, $d = 0.5$ m; (f) $H_m = H_e = 4\ \mathrm{m}^{-1}$; $d = 2$ m.

$$Q = \frac{4}{a} k_s \sum_{n=1}^{+\infty} \left[ (H_m - H_e) \frac{\sin \mu_n c}{\mu_n} - H_e \frac{(-1)^n}{\mu_n} \right] \left( C_n + \frac{(-1)^n}{\mu_n} T_i \right). \quad (10)$$

Figure 5 shows the dependence of the total heat loss on both width and thermal $U$-value of the insulation. The slab configuration is similar to that of Fig. 4. As expected, Fig. 5 indicates that an increase in outer perimeter insulation (either through increasing the width $d$ or decreasing the $H_a$ value) reduces the total heat loss from the slab floor. This decrease however is subject to the law of diminishing returns. Over half of the slab heat loss reduction is achieved in the first 0.5 m of the outer insulation. For this first 0.5 m of




(g)




(h)

FIG. 3.—*Continued.* (g) $H_m = H_e = 4\ m^{-1}$, $H_a = 0.1\ m^{-1}$, $d = 4$ m; (h) $H_m = 4\ m^{-1}$, $H_e = H_a = 0.1\ m^{-1}$, $c = d = 2$ m.

outer insulation, the slab heat loss reductions do not exhibit a strong variation with the insulation $U$-value. In general, the impact of outer insulation width on slab heat loss is more significant than that of the insulation $U$-value.

Figure 6 illustrates the effect of both water table depth and the inner edge insulation location (i.e. the parameter $c$) on the total slab heat loss. The insulation $H$-value is assumed to be $H_c = 0.5\ m^{-1}$. For a given water table depth, the increase in the inner edge insulation width (i.e. the parameter $(a-c)$) reduces the total heat loss from the slab following the law of diminishing returns. For a given insulation configuration, Fig. 6 shows that the magnitude of total slab heat loss increases as the water table level decreases. In addition, the rate of reduction in slab

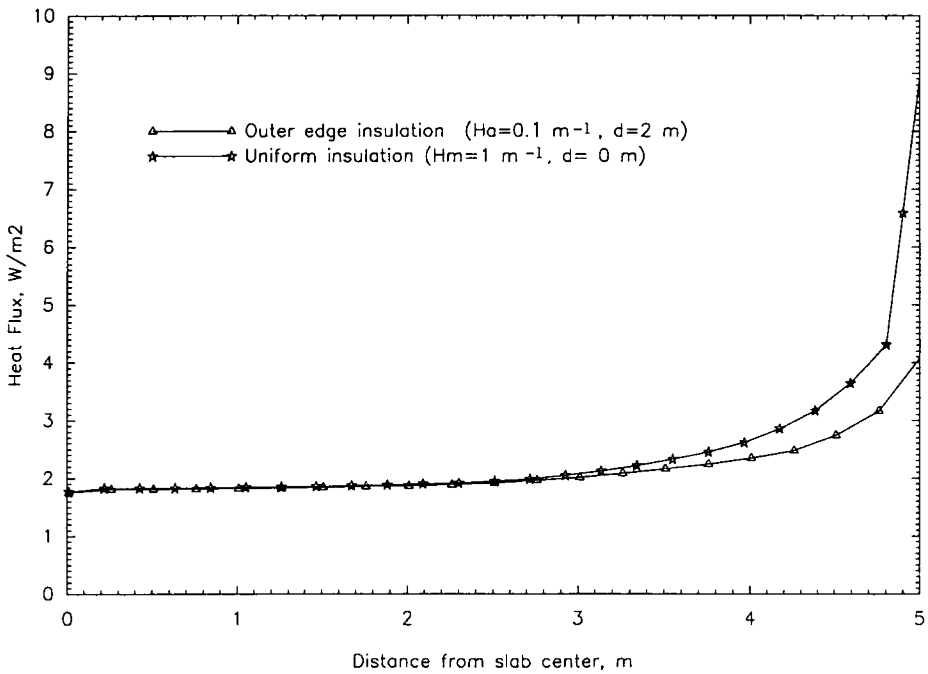



FIG. 4. Effect of outer edge insulation on local heat flux distribution along an uninsulated slab-on-grade floor.

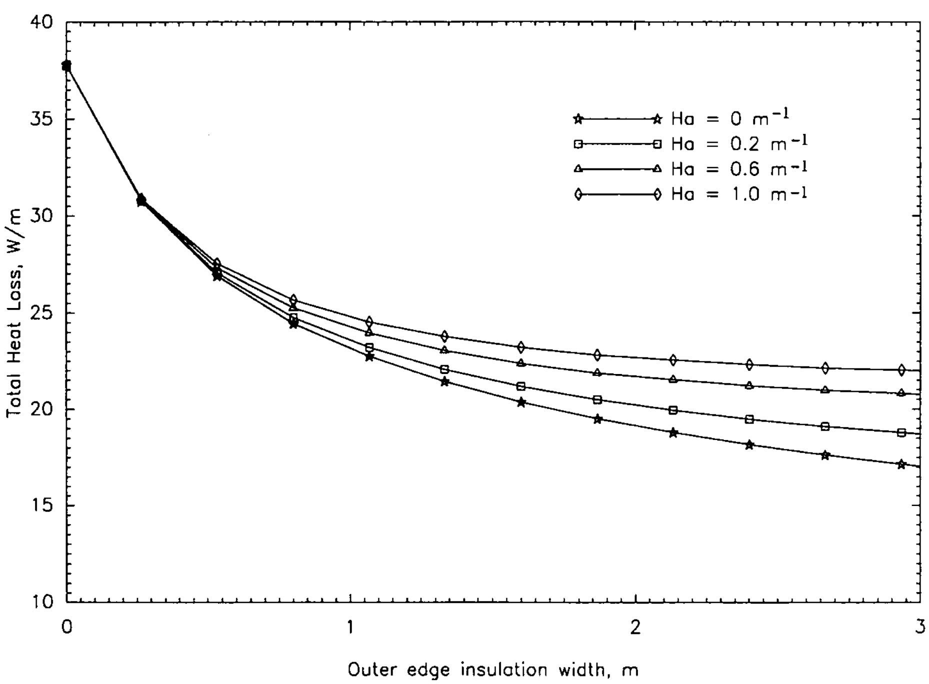



FIG. 5. Effect of outer edge insulation and its length on the total slab heat loss.

heat loss due to an increase in the inner edge insulation width increases as the water table depth decreases.

## 6. CONCLUSION

The ITPE technique is applied to develop the steady-state solution of the heat conduction equation beneath a slab with horizontal insulation. The insulation is placed under the slab and/or at the ground surface outside the slab. The effect of the insulation width and $U$-value on foundation heat losses is analyzed. In particular, it is shown that outer insulation is effective in reducing heat loss from slab edges. It is found in particular that it is thermally better to

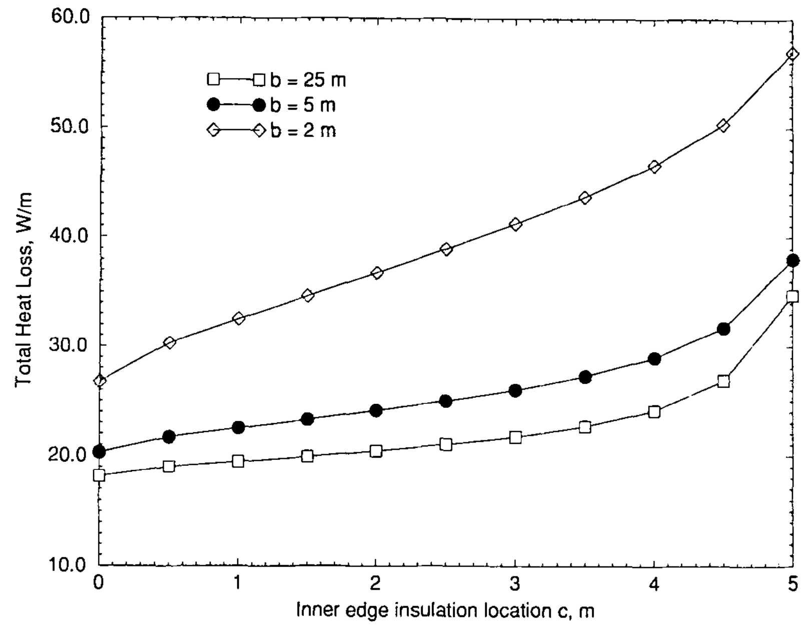



FIG. 6. Effect of water table level on the total heat loss from a slab with inner edge insulation.

extend the length of outer edge insulation rather than increase the insulation thermal resistance over a short distance from the slab edge. Finally, the total slab heat loss was found to be significantly affected by the water table level.

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